Anomalous acoustic behaviour and backbone structure of percolation clusters

1984 J. Phys. A: Math. Gen. 17 L137

(http://iopscience.iop.org/0305-4470/17/3/008)

J. Phys. A: Math. Gen. 17 (1984) L137–L142. Printed in Great Britain

LETTER TO THE EDITOR

Anomalous acoustic behaviour and backbone structure of percolation clusters

T Ohtsuki and T Keyes
Department of Chemistry, Boston University, Boston, Massachusetts 02215, USA

Received 24 October 1983

Abstract. A real-space renormalisation group method is used to investigate acoustic properties of percolation lattices in the vicinity of the percolation threshold. Scaling form and expressions for critical exponents x, y describing the sound velocity are derived. The existence of an anomalous dispersion relation is elucidated. Dimensionalities d_T and d_L which represent the geometrical structure of cluster backbones are introduced. The exponents x, y and the dimensionalities d_T, d_L are expressed in terms of other exponents ν, β, t, s and β_B for percolation and their explicit values are evaluated from known estimates for ν, β, t, s and β_B. A geometrical picture of the backbone structure is obtained in terms of the dimensionalities.

Dynamical properties of percolation lattices are a subject of rapidly growing interest. The fractal and dendritic structure of percolation clusters causes various types of anomalous behaviour near the percolation threshold p_c. For example, the mean-square displacement $\langle R^2(t)\rangle$ at the threshold $p=p_c$ varies as $\langle R^2(t)\rangle \propto t^\chi$ where χ is a critical exponent less than unity (de Gennes 1976a, Ben-Avraham and Havlin 1982, Gefen *et al* 1983). Recently Alexander and Orbach (1982) reported that this anomalous diffusion yields the fractal (fracton) dimensionality for the density of states (see also Rammal and Toulouse 1983). Harris and Stinchcombe (1983) derived anomalous dispersion relations and dynamic critical exponents for dilute ferromagnets. These facts suggest that the lattice vibrations of percolation lattices also show anomalous behaviour. In spite of both theoretical and practical importance, however, little is known about the acoustic behaviour of percolation clusters.

On the other hand, the understanding of cluster structure is indispensable to study not only acoustic but also magnetic properties of dilute ferromagnets, the conductivity of random resistor networks etc. Since only the backbone of the infinite cluster is responsible for some of these properties, knowledge of the backbone structure is of particular significance. Various pictures have been proposed in the literature (Skal and Shklovskii 1974, de Gennes 1976b, Stanley 1977, Kirkpatrick 1978, Coniglio 1982).

The purpose of this letter is twofold: first, to investigate acoustic properties of percolation lattices such as sound velocity and dispersion relation, in particular, their critical behaviour in the vicinity of the percolation threshold, and second, to calculate the dimensionalities for cluster backbones and clarify their geometrical structure.

First, we derive the scaling form and expressions of critical exponents x, y for the sound velocity c, on the basis of a real-space renormalisation group (RSRG) method which is frequently useful for percolation problems (Stanley *et al* 1982, Ohtsuki and

0305-4470/84/030137+06\$02.25

Keyes 1983). Then an anomalous dispersion relation is obtained. Dimensionalities d_T and d_L are introduced and their geometrical meaning is given. Lastly, we express x, y, d_T and d_L in terms of other critical exponents ν, β, t, s and β_B (Stauffer 1979, Straley 1978, Kirkpatrick 1978, Stanley 1977).

We consider sound propagation on a harmonic lattice with lattice constant l, mass M_i of the ith particle and force constant Γ_{ij} between the nearest-neighbour pair i, j. Here two classes of bond percolation problems are treated, that is, $M_i = m$ for all i and Γ_{ij} is an independent random variable with a binary probability distribution

$$\text{(a)} \qquad P_a(\Gamma_{ij}) = p\delta(\Gamma_{ij} - \gamma) + (1-p)\delta(\Gamma_{ij}), \tag{1}$$

$$\text{(b)} \qquad P_b(\Gamma_{ij}^{-1}) = p\delta(\Gamma_{ij}^{-1}) + (1-p)\delta(\Gamma_{ij}^{-1} - \gamma^{-1}). \tag{2}$$

Case (a) is related to the problem of the conductivity of a random resistor network composed of conductor and insulator bonds and (b) to that composed of superconductor and conductor bonds. Case (a) corresponds to a solid disordered by vacancies and might be a good model just below the melting transition. Case (b) might represent a liquid with solid-like 'icebergs'. We expect the speed of sound c to vanish (diverge) as p_c is approached from above (below) in case (a) ((b)) and only consider case (a) ((b)) for $p \geq p_c$ ($p \leq p_c$).

In both cases, we apply the RSRG method in the usual way (Stanley *et al* 1982). Groups of sites and bonds are combined into supersites and bonds which form a new renormalised lattice with lattice constant

$$l' = bl. \tag{3}$$

Under the requirement that acoustic properties of the lattices are kept invariant, we assume the existence of the renormalisation transformation such that on the new lattice, the mass M_i' of a particle is the same for all particles $M_i' = m'$ and there is only the nearest-neighbour force whose force constant Γ_{ij}' is also an independent random variable with a binary probability distribution

$$\text{(a)} \qquad P_a'(\Gamma_{ij}') = p'\delta(\Gamma_{ij}' - \gamma') + (1-p')\delta(\Gamma_{ij}'), \tag{4}$$

$$\text{(b)} \qquad P_b'(\Gamma_{ij}'^{-1}) = p'\delta(\Gamma_{ij}'^{-1}) + (1+p')\delta(\Gamma_{ij}'^{-1} - \gamma'^{-1}). \tag{5}$$

Generally the existence of such a transformation is not obvious, but near the percolation threshold it is thought to be verified by the self-similarity and the scale invariance of percolation clusters (Kapitulnik *et al* 1983). The dimensional analysis, namely the Π-theorem, gives recursion relations of the form

$$p' = w(p, b), \tag{6}$$

$$\gamma' = \gamma b^{d-2} f(p, b), \tag{7}$$

$$m' = mb^d h(p, b), \tag{8}$$

where w, f and h are smooth (once differentiable) functions of p and b normalised so that $w = f = h = 1$ at $p = 1$ and d is the Euclidean dimensionality of the system. Similarly, we get

$$c(m, \gamma, l, p; k) = (\gamma/m)^{1/2} lc^*(p, lk), \tag{9}$$

where k is a wavevector and c^* is the normalised dimensionless sound velocity. Substituting (3) and (6)–(8) into (9) and the requirement

$$c(m, \gamma, l, p; k) = c(m', \gamma', l', p'; k), \tag{10}$$

we obtain the recursion relation for c^*

$$c^*(p, K) = [f(p, b)/h(p, b)]^{1/2} c^*(w(p, b), bK), \tag{11}$$

where $K \equiv lk$ is the normalised dimensionless wavevector.

In the vicinity of the percolation threshold as $|\varepsilon| = |p - p_c|/p_c \ll 1$, the recursion relation (11) gives the scaling form of c^* and the expressions of accompanying critical exponents. In this region, (6)–(8) are expressed as

$$p' - p_c = (p - p_c) w'_c + O(\varepsilon^2), \tag{12}$$

$$f = f_c + O(\varepsilon), \tag{13}$$

$$h = h_c + O(\varepsilon), \tag{14}$$

where $w' = \partial w/\partial p$ and the subscript c denotes the quantity at $p = p_c$. Substitution of these equations in (11) leads to

$$c^*(\varepsilon, K) \simeq (f_c/h_c)^{1/2} c^*(\varepsilon w'_c, bK). \tag{15}$$

Iterating this procedure n times and putting $\delta = \varepsilon(w'_c)^n$, we have

$$c^*(\varepsilon, K)/|\varepsilon|^x = c^*(\delta, K/|\delta/\varepsilon|^{-\nu})/|\delta|^x \tag{16}$$

where x is the critical exponent defined by

$$x \equiv \tfrac{1}{2}\ln(h_c/f_c)/\ln w'_c \tag{17}$$

and ν is the exponent for the coherence length ξ defined by $\xi \propto |\varepsilon|^{-\nu}$ and given by $\nu = \ln b/\ln w'_c$ (Stauffer 1979). Equation (16) is valid for any ε, $\delta(n)$ and K such that $1 \gg |\delta| > |\varepsilon| \neq 0$. Thus we can derive the scaling form of c^*

$$c^*(\varepsilon, K) = |\varepsilon|^x F_\pm(K|\varepsilon|^{-\nu}) = \xi^{-x/\nu} F_\pm(K\xi), \tag{18}$$

where subscript + or − represents above or below the threshold and corresponds to case (a) or (b).

The limit of (15) when $K \to 0$ or $\varepsilon \to 0$ yields

$$c^*(\varepsilon, 0) \propto |\varepsilon|^x, \tag{19}$$

$$c^*(p_c, K) \propto K^y, \tag{20}$$

with the critical exponent y given by

$$y \equiv \tfrac{1}{2}\ln(h_c/f_c)/\ln b = x/\nu. \tag{21}$$

Combining (18) with (19) and (20), we find that in the limit $Z \to 0$ or $Z \to \infty$, $F_\pm(Z)$ is proportional to Z^0 or Z^y, respectively. Then we have

$$c^*(\varepsilon, K) \propto |\varepsilon|^x K^0 \qquad (K \ll \xi^{-1}), \tag{22}$$

$$c^*(\varepsilon, K) \propto \varepsilon^0 K^y \qquad (\xi^{-1} \ll K \ll 1). \tag{23}$$

Equation (23) leads to the anomalous dispersion relation

$$\Omega = c^* K \propto K^{1+y} \qquad (\xi^{-1} \ll K \ll 1) \tag{24}$$

where $\Omega \equiv \omega/\omega_0 \equiv \omega(m/\gamma)^{1/2}$ is the normalised dimensionless frequency. This relation is also derived as follows. Let c_f be the frequency-dependent sound velocity. As a result of the dimensional analysis, we have

$$c_f(m, \gamma, l, p; \omega) = (\gamma/m)^{1/2} l c_f^*(p, (m/\gamma)^{1/2}\omega). \tag{25}$$

Since a similar relationship to (10) also exists for c_f, we obtain the recursion relation for c_f^*,

$$c_f^*(p,\Omega)=[f(p,b)/h(p,b)]^{1/2}c_f^*(w(p,b),[h(p,b)/f(p,b)]^{1/2}b\Omega). \quad (26)$$

At $p=p_c$, equation (26) yields

$$c_f^*(p_c,\Omega)\propto\Omega^{y/(1+y)}. \quad (27)$$

Comparing (20) and (27), we find the dispersion relation (24), because c^* and c_f^* represent the same physical quantity.

As mentioned before, investigation of the backbone structure of percolation clusters has a significant meaning. To this end, we here consider acoustic properties of only cluster backbones without dead ends. Concretely, we treat the percolation lattice in case (a) where bonds with a force constant γ belonging to dead ends are replaced by those with zero force constant. After the replacement, the value of the probability p alters but physical quantities are still functions of the initial value, because the replacement is a univalent operation. Since the cluster backbone itself is also self-similar (Kirkpatrick 1978), the preceding discussions are considered to hold for this case. That is, the normalised sound velocity c_B^* varies as

$$c_B^*(\varepsilon,K)\propto|\varepsilon|^{x_B}K^0 \qquad (K\ll\xi^{-1}), \quad (28)$$

$$c_B^*(\varepsilon,K)\propto\varepsilon^0K^{y_B} \qquad (\xi^{-1}\ll K\ll 1), \quad (29)$$

with $y_B=x_B/\nu$ and the dispersion relation is given by

$$\Omega=c_B^*K\propto K^{1+y_B} \qquad (\xi^{-1}\ll K\ll 1). \quad (30)$$

In normal d-dimensional lattices, the number of modes $N(K)$ with wavevectors less than K satisfies $N(K)\propto K^d$. In the low frequency limit, the sound velocity usually becomes independent of K and the frequency spectrum $g(\Omega)$ is given by

$$g(\Omega)=\mathrm{d}N_f(\Omega)/\mathrm{d}\Omega\propto\Omega^{d-1} \quad (31)$$

where $N_f(\Omega)=N(\Omega/c^*)$ is the number of modes with frequencies less than Ω. For $K\ll\xi^{-1}$, namely, $\Omega\ll c_B^*(\varepsilon,0)\xi^{-1}\simeq\xi^{-(1+y_B)}$, equation (31) is applicable and we have

$$g_B(\Omega)\propto\Omega^{d-1} \qquad (\Omega\ll\xi^{-(1+y_B)}). \quad (32)$$

In contrast, when $\xi^{-1}\ll K\ll 1$ and $\xi^{-(1+y_B)}\ll\Omega\ll 1$, the fractal structure of cluster backbones and the anomalous dispersion relation lead to an anomalous frequency spectrum. Since modes with finite velocity are generated only on the cluster backbone, we get

$$N(K)\propto K^{d_B} \quad (33)$$

where $d_B=d-\beta_B/\nu$ is the fractal dimensionality of the cluster backbones (Kirkpatrick 1978). From (30), it follows that

$$g_B(\Omega)\propto\Omega^{d_T-1} \qquad (\zeta^{-(1+y_B)}\ll\Omega\ll 1) \quad (34)$$

with the dimensionality d_T defined by

$$d_T\equiv d_B/(1+y_B). \quad (35)$$

The geometrical meaning of the dimensionality d_T is considered as follows. In general, cluster backbones are not one-dimensional chains but composed of 'links' and 'blobs' (Stanley 1977, Coniglio 1982). The density of states only on cluster backbones at $1 \gg K \gg \xi^{-1}$ is just that of these links and blobs. Thus we suggest that the dimensionality d_T represents the effective 'thickness' of links and blobs, i.e. backbones. On the other hand, the line along links and blobs describes an irregular curve with fractal dimension. Since the dimensionality d_B contains both effects (thickness and irregularity), we put

$$d_B = d_T d_L, \qquad (36)$$

where d_L is the fractal dimensionality for the effective 'length' of backbones along links and blobs. This decomposition of d_B into d_T and d_L is thought to be useful to make the geometrical structure of cluster backbones clearer.

We now relate the critical exponents x, y and the dimensionality d_T with other critical exponents for percolation (Stauffer 1979, Straley 1978). In the low-frequency limit, the sound velocity is proportional to the square root of the elastic modulus E of the lattice and inversely proportional to the square root of the density ρ of the system. Near the percolation threshold, E is considered to vary in the same way as the conductivity σ of the corresponding resistor network (de Gennes 1976b). In case (a), only the infinite cluster vibrates and ρ satisfies $\rho \propto \varepsilon^{\beta}$. Similarly, we have $\rho \propto \varepsilon^{\beta_B}$ for c_B, while in case (b), ρ has no singularity at the threshold. Since dead ends have no contribution to E, we obtain

$$c \propto \varepsilon^{(t-\beta)/2} \qquad (\text{case (a)}, \varepsilon > 0), \qquad (37)$$

$$c \propto |\varepsilon|^{-s/2} \qquad (\text{case (b)}, \varepsilon < 0), \qquad (38)$$

$$c_B \propto \varepsilon^{(t-\beta_B)/2}, \qquad (39)$$

where t and s are the critical exponents for the conductivity σ of the conductor–insulator network and that of the superconductor–conductor network, respectively. The critical exponents x and y are expressed as

$$x = \tfrac{1}{2}(t-\beta), \qquad y = (t-\beta)/2\nu \qquad (\text{case (a)}), \qquad (40)$$

$$\bar{x} = -\tfrac{1}{2}s, \qquad \bar{y} = -s/2\nu \qquad (\text{case (b)}), \qquad (41)$$

$$x_B = \tfrac{1}{2}(t-\beta_B), \qquad y_B = (t-\beta_B)/2\nu, \qquad (42)$$

and the dimensionality d_T is written as

$$d_T = d_B/[1 + (t-\beta_B)/2\nu]. \qquad (43)$$

The exponent $1 + y = 1 + (t-\beta)/2\nu$ for the anomalous dispersion relation is just half of that for dilute ferromagnets derived by Harris and Stinchcombe (1983).

The dimensionality d_T corresponds to the fracton dimensionality $\bar{d}$ introduced by Alexander and Orbach (1982). They considered diffusion on clusters and then derived $\bar{d}$. The mean-square displacement $\langle S^2 \rangle$ along links and blobs satisfies $\langle S^2 \rangle \propto t$, irrespective of thickness d_T, and the mean-square displacement $\langle R^2 \rangle$ in the Euclidean space is related to $\langle S^2 \rangle$ as $\langle R^2 \rangle^{d_L} \propto \langle S^2 \rangle$, namely $\langle R^2 \rangle \propto t^{1/d_L}$. Since $\bar{d} = d_B$ in this case, their expression for $\bar{d}$ becomes equivalent to (36). Equation (43) shows that the critical exponent for the diffusion coefficient D_B only on the backbone of the infinite cluster is given by $t - \beta_B$. However, it should be noted that $t - \beta_B$ is the exponent for D_B and is different from that for the diffusion coefficient on the whole cluster, where dead ends also play some roles (Ohtsuki and Keyes 1983).

By using known estimates for ν, β, t, s and β_B (Stauffer 1979, Straley 1978, Kirkpatrick 1978, Stanley 1977), we can calculate explicit values of the exponents x, y and the dimensionalities d_T, d_L. The results are listed in table 1.

Table 1. Critical exponents and dimensionalities.

d	ν[a]	β[a]	t[a]	s[b]	β_B[c]	x	y	$\bar{x}$	$\bar{y}$	d_B	d_T	d_L
1	1.0	0		1.0	0			−0.5	−0.5	1.0	1.0	1.0
2	1.35	0.14	1.2	1.1	0.5	0.5	0.4	−0.55	−0.4	1.6	1.3	1.3
3	0.84	0.4	1.7	0.7	0.9	0.65	0.8	−0.35	−0.4	2.0	1.3	1.5
4	0.7	0.5	2.4	0.6	1.1	0.95	1.4	−0.3	−0.4	2.4	1.3	1.9
5	0.6	0.7	2.7									
6	0.5	1.0	3.0	0		1.0	2.0	0	0	2.0[d]	1.0	2.0

[a] Stauffer (1979).
[b] Straley (1978).
[c] Kirkpatrick (1978).
[d] Stanley (1977).

Above six dimensions, the dimensionality d_T is equal to unity and d_L is equal to two. These values are compatible with the picture proposed by Skal and Shklovskii (1974) for random resistor networks, i.e. one-dimensional channels with crosslinks, because in these dimensions, excluded volume effects are also negligible (de Gennes 1972) and one-dimensional channels should make a pure random walk. It should be emphasised that for $1 < d < 6$, however, their picture is not applicable. At $2 \leq d \leq 4$, d_T is about 1.3 and more than unity. On the other hand, d_L is nearly equal to that for the self-avoiding walk of one-dimensional chains (de Gennes 1972). These results support a nodes, links and blobs model of cluster backbones proposed recently instead of the simple nodes and links model (Stanley 1977, Coniglio 1982).

This work was partly supported by the NSF, grant No CHE81-11422, by a Dreyfus Teacher–Scholar grant and by the Petroleum Research Fund.

References

Alexander S and Orbach R 1982 *J. Physique Lett.* **43** L625
Ben-Avraham D and Havlin S 1982 *J. Phys. A: Math. Gen.* **15** L691
Coniglio A 1982 *J. Phys. A: Math. Gen.* **15** 3829
de Gennes P G 1972 *Phys. Lett.* **38A** 339
—— 1976a *La Recherche* **7** 919
—— 1976b *J. Physique Lett.* **37** L1
Gefen Y, Aharony A and Alexander S 1983 *Phys. Rev. Lett.* **50** 77
Harris C K and Stinchcombe R B 1983 *Phys. Rev. Lett.* **50** 1399
Kapitulnik A, Aharony A, Deutscher G and Stauffer D 1983 *J. Phys. A: Math. Gen.* **16** L269
Kirkpatrick S 1978 *AIP Conf. Proc.* **40** 99
Ohtsuki T and Keyes T 1983 submitted
Rammal R and Toulouse G 1983 *J. Physique Lett.* **44** L13
Skal A and Shklovskii B I 1974 *Fiz. Tekh. Poluprov.* **8** 1586 (1975 *Sov. Phys.–Semicond.* **8** 1029)
Stanley H E 1977 *J. Phys. A: Math. Gen.* **10** L211
Stanley H E, Reynolds P J, Redner S and Family F 1982 in *Real-Space Renormalization* ed T W Burkhardt and J M J van Leeuwen (Berlin: Springer)
Stauffer D 1979 *Phys. Rep.* **54** 3
Straley J P 1978 *AIP Conf. Proc.* **40** 118